In this talk, we will introduce the uniform Roe algebras of uniform locally finite metric spaces. We will discuss some examples of these algebras and examine their basic properties. We conclude the series of talks with a brief introduction to the rigidity problem of uniform Roe algebras.

3 Operator algebras: uniform Roe algebras of uniformly locally finite spaces

Recall from the first talk that a metric space (X, d) is uniformly locally finite (ulf for short) if for every $R \geq 0$ there exists a number $N \in \mathbb{N}$ such that for every $x \in X$ one has

$$|B_R(x)| \leq N.$$

Concretely, ulf spaces are the metric spaces whose R-balls have a finite amount of elements in a uniform way. Also recall that by Rieffel's theorem, two unital (more generally σ-unital) C^*-algebras are Morita-equivalent if and only if they are stabely isomorphic

$$A \sim_{\text{Morita}} B \iff A \otimes \mathbb{K} \cong B \otimes \mathbb{K},$$

where $\mathbb{K}$ is the algebra of compact operators on a separable infinite dimensional Hilbert space. We want to find an operator algebraic invariant of (bijective) coarse equivalences for uniformly locally finite coarse spaces. Let us start with an example of finitely generated groups. When given a finitely generated group, one of the first C^*-algebras that comes to mind is the reduced group C^*-algebra.

3.1 (Reduced group C^*-algebras)**.** Let G be a finitely generated group, and let $C_r^*(G)$ be the reduced C^*-algebra of G, i.e. a C^*-algebra generated by unitaries

$$\lambda_g \colon \ell^2(G) \to \ell^2(G), \quad \lambda_g(\delta_h) = \delta_{gh}, \quad g, h \in G.$$

Note that the isomorphism class of $C_r^*(G)$ is not a coarse invariant (for instance, all finite groups are coarsely equivalent, but the reduced group C^*-algebras have different dimensions). It is not even a bijective coarse invariant, since $\mathbb{Z}$ and D_∞ (the infinite Dihedral group) are bijectively coarsely equivalent (as they are biLipschitz equivalent), but $C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$ and $C_r^*(D_\infty)$ is non-commutative. Hence, to get a bijective coarse invariant, we have to modify the reduced group C^*-algebra.

Remark 3.2. One may check that $C_r^*(D_\infty)$ is isomorphic to a C^*-subalgebra of $C([0,1]) \otimes M_2(\mathbb{C})$ given by functions that diagonalise on the ends of the interval. This isomorphism is induced by an injective $*$-homomorphism

$$\Phi \colon C_r^*(D_\infty) \to C([0,1]) \otimes M_2(\mathbb{C}), \quad \lambda_u \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda_v \mapsto \begin{pmatrix} 1-2x & 2\sqrt{x(1-x)} \\ 2\sqrt{x(1-x)} & 2x-1 \end{pmatrix}$$

To see that Φ is well-defined note that since D_∞ is amenable one has $C^*(D_\infty) \cong C_r^*(D_\infty)$, and since $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ the C^*-algebra $C_r^*(D_\infty)$ is a universal C^*-algebra generated by two unitaries of order 2. Hence the $*$-homomorphism Φ is well-defined. One may also check that it is injective and surjects onto the aforementioned subalgebra. Note that the K_0-group of $C(\mathbb{T})$ is isomorphic to $\mathbb{Z}$, since $C(\mathbb{T})$ is the unitalization of $C_0(\mathbb{R})$, whose K_0-group is zero. The K_0-group of $C_r^*(D_\infty)$ may be computed as follows. Note that there is an exact sequence

$$0 \to C_0(\mathbb{R}) \otimes M_2(\mathbb{C}) \to C_r^*(D_\infty) \to \mathbb{C}^4 \to 0$$

Hence, by the 6-term exact sequence in K-theory, one has the exactness of the following sequence

$$0 \to K_0(C_r^*(D_\infty)) \to \mathbb{Z}^4 \to \mathbb{Z}.$$

It follows that $K_0(D_\infty)$ cannot equal to $\mathbb{Z}$. In particular, $C_r^*(D_\infty)$ and $C_r^*(\mathbb{Z})$ are not Morita equivalent.

The first guess was not successful. One of the reasons for it is the fact that the reduced group C^*-algebra tracks the commutativity of the underlying group. Let's try something commutative.

3.3 (Bounded functions)**.** Let G be a finitely generated group, and let $\ell^\infty(G)$ be a C^*-algebra of all bounded functions from G to $\mathbb{C}$. We can represent $\ell^\infty(G)$ on $\ell^2(G)$ by declaring

$$\pi(f)v = fv, \qquad v \in \ell^2(G), \;\; f \in \ell^\infty(G).$$

It is easy to see that this representation is faithful (it has no kernel). Given two bijectively coarsely equivalent groups G and H, we can define a unitary operator

$$U_f \colon l^2(G) \to l^2(H), \quad \delta_g \mapsto \delta_{f(g)},$$

for every bijective coarse equivalence f. Note that the conjugation by U_f maps $\ell^\infty(G)$ to $\ell^\infty(H)$:

$$U_f \pi_G(g) U_f^*(\delta_h) = g(f^{-1}(h))\delta_h, \quad g \in \ell^\infty(G)$$

Hence every bijective coarse equivalence $f\colon G \to H$ induces a $*$-isomorphism $\mathrm{Ad}_{U_f}\colon \ell^\infty(G) \to \ell^\infty(H)$, and $\ell^\infty(G)$ is an invariant of bijective coarse equivalence.

Despite the fact that $\ell^\infty(G)$ is an invariant of bijective coarse equivalence, this invariant is not very interesting. We didn't use the metrics on G and H, so the same procedure can be done for any bijection $f\colon G \to H$.

Remark 3.4. One may ask whether a coarse equivalence induce $f\colon G \to H$ induce a Morita equivalence between $\ell^\infty(G)$ and $\ell^\infty(H)$, and the answer is again negative. Consider $G = \mathbb{Z}/2\mathbb{Z}$ and $H = \mathbb{Z}/3\mathbb{Z}$, then

$$\ell^\infty(G) = \mathbb{C}^2, \quad \ell^\infty(H) = \mathbb{C}^3,$$

hence by additivity of K_0 we get that the respective K_0-groups are $\mathbb{Z}^2$ and $\mathbb{Z}^3$.

To get a better invariant, we have to modify the algebra of bounded functions on G. It turns out that the combination of the above two algebras provides a well-behaved invariant. It is, however, easier to deal with more general spaces than with groups. Hence, we will first develop the general theory, and then we will come back to the group case.
Suppose given a uniformly locally finite metric space (X, d). Consider a Hilbert space $\ell^2(X)$ of square summable functions from X to $\mathbb{C}$. One can view operators on $\ell^2(X)$ as X by X matrices via the identification

$$T \mapsto [\langle T\delta_x, \delta_y\rangle]_{x,y\in X},$$

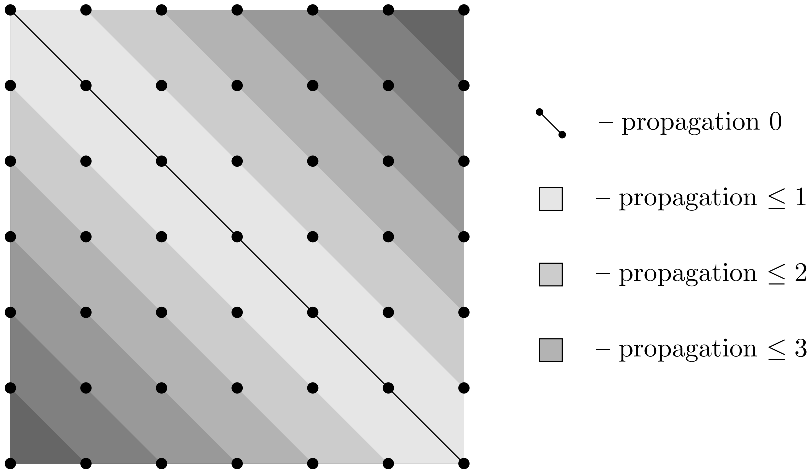
where δ_x is a delta-function at a point x. Note that not every X by X matrix defines a bounded operator, for instance, if $a_{x,y} = \delta_{x_0}(x)$, for some point $x_0 \in X$, then the matrix $[a_{x,y}]_{x,y\in X}$ is not a bounded operator. We will extract a certain concrete C^*-subalgebra of $\mathcal{B}(\ell^2(X))$.

Definition 3.5 (Propagation)**.** Let (X, d) be a ulf metric space, and $T \in \mathcal{B}(\ell^2(X))$ be a bounded operator. The propagation of T is the following quantity:

$$\mathrm{prop}(T) = \sup\{d(x, y) \mid x, y \in X; \;\; \langle T\delta_x, \delta_y\rangle \neq 0\}.$$

A bounded operator T is said to have finite propagation if $\mathrm{prop}(T) < \infty$.

The propagation of an operator can be easily visualised. Consider, for example, a ulf space $(\mathbb{Z}, d)$, where d is the Euclidean metric. Then, every bounded operator can be represented by an infinite $\mathbb{Z} \times \mathbb{Z}$ matrix. One can think of it as an infinite Excel table with complex entries. Suppose that $\mathrm{prop}(T) = 0$, then $\langle T\delta_n, \delta_m\rangle \neq 0$ if and only if $n = m$, hence the only nonzero entries of the table are the diagonal ones. It follows that $T \in \ell^\infty(\mathbb{Z})$ (the same argument works for any ulf space X). Suppose that $\mathrm{prop}(T) \leq N$, then $\langle T\delta_n, \delta_m\rangle \neq 0$ if and only if $d(n, m) \leq N$, hence the nonzero entries of the table are located in a strip (tube) around the diagonal of radious N. This can be illustrated by the following picture:



Finally, we provide an example of an operator with infinite propagation. Consider an operator $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$ defined by

$$T\delta_k = \begin{cases} \delta_{2k}, & \text{if } k \text{ is a power of 2;} \\ 0 & \text{otherwise.} \end{cases}$$

By definition of propagation of has $\mathrm{prop}(T) = \sup_k |2k - k| = \infty$. Hence, T has infinite propagation. Moreover, for every finite propagation operator $T' \in \mathcal{B}(\ell^2(\mathbb{Z}))$ one has $\|T - T'\| > 1$. Note that operators of finite propagation need not form a closed subset of $\mathcal{B}(\ell^2(X))$.

Definition 3.6 (The uniform Roe algebra)**.** Let (X, d) be a ulf space. The uniform Roe algebra $C^*_u(X)$ is the closure of finite propagation operators in $\mathcal{B}(\ell^2(X))$.

As we mentioned before, the diagonal operators $\ell^\infty(X)$ have propagation 0, therefore $\ell^\infty(X) \subset C^*_u(X)$. Moreover, since $\ell^\infty(X)$ is a maximal abelian subalgebra (MASA) in $\mathcal{B}(\ell^2(X))$ it is a MASA of $C^*_u(X)$, and the canonical conditional expectation $E\colon \mathcal{B}(\ell^2(X)) \to \ell^\infty(X)$ restricts to a canonical conditional expectiation on $C^*_u(X)$. Note also that every operator $T \in \mathcal{B}(\ell^2(X))$ of the form

$$T_{x,y}v = \langle v, \delta_y \rangle \, \delta_x, \quad x, y \in X$$

has finite propagation (since $\mathrm{prop}(T_{x,y}) = d(x, y)$ for all pints x and y), therefore the. closure of the linear span of such operators is contained in $C^*_u(X)$. Since the closure generates the C^*-algebra of compact operators on $\ell^2(X)$, we have the inclusion $\mathbb{K}(\ell^2(X)) \subset C^*_u(X)$.

3.7 (Uniform Roe algebra of a group)**.** Let (G, d) be a finetely generated group with a right metric induced from a Cayley graph of G. For $g \in G$ note that

$$\mathrm{prop}(\lambda_g) = \sup\{d(h, k) \mid \langle \lambda_g \delta_h, \delta_k \rangle \neq 0\} = \sup\{d(h, gh) \mid h \in G\} = \gamma(g) < \infty.$$

Hence, λ_g has finite propagation, and we have an inclusion $C^*_r(G) \subset C^*_u(G)$.

As announced at the beginning, the combination of the two unsuccessful attempts leads to the successful one. The following lemma makes this statement precise.

Lemma 3.8. The uniform Roe algebra of a finitely generated group equipped with a right invariant metric from its Cayley graph is generated by $\ell^\infty(G)$ and $C^*_r(G)$.

Proof. As mentioned above the union $\ell^\infty(G) \cup C^*_r(G)$ is contained in $C^*_u(G)$, therefore one inclusion is established. For the second inclusion, consider a finite propagation operator $T \in \mathcal{B}(\ell^2(G))$, whose propagation is at most N. Then, all nonzero entries of the matrix representation of T are located in the strip

$$E_N = \{(g, h) \mid \gamma(gh^{-1}) \leq N\}.$$

Denote by $\Sigma \subset G$ a finite set of all elements $g \in G$ that satisfy $\gamma(g) \leq N$, then one can rewrite the definition of E_N as follows

$$E_N = \{(g, h) \mid gh^{-1} \in \Sigma\} = \bigsqcup_{\sigma \in \Sigma} E^\sigma,$$

where $E^\sigma = \{(g,h) \mid gh^{-1} = \sigma\}$. It follows that T can be decomposed as

$$T = \sum_{\sigma\in\Sigma} a_\sigma \lambda_\sigma, \quad a_\sigma \in \ell^\infty(G) \text{ for all } \sigma \in \Sigma.$$

Hence, finite propagation operators are contained in the C^*-algebra generated by the union $\ell^\infty(G) \cup C_r^*(G)$. This finishes the proof. □

Remark 3.9. One may prove that for a finitely generated group G the aforementioned C^*-algebra $C_u^*(G)$ is isomorphic to the reduced crossed product $\ell^\infty(G) \rtimes_r G$. Note that this C^*-algebra is not separable when G is infinite, as $\ell^\infty(G)$ is not separable.

We are now to establish the fact that $C_u^*(X)$ is an invariant under bijective coarse equivalences. Suppose for two metric spaces (X, d), (Y, ∂) given a bijective coarse equivalence $f\colon X \to Y$. Recall the unitary U_f defined during the second unsuccessful attempt

$$U_f\colon \ell^2(X) \to \ell^2(Y), \quad \delta_x \mapsto \delta_{f(x)}.$$

Now $\mathrm{Ad}_{U_f}\colon \mathcal{B}(\ell^2(X)) \to \mathcal{B}(\ell^2(Y))$ is a $*$-isomorphism, since f is a bijection. Moreover, for a finite propagation operator T (say of propagation N), one has

$$\mathrm{prop}(\mathrm{Ad}_{U_f}(T)) = \sup\{\partial(x,y) \mid \langle TU_f^*\delta_x, U_f^*\delta_y\rangle \neq 0\} = \sup\{\partial(x,y) \mid d(f^{-1}(x), f^{-1}(y)) \leq N\} < \infty,$$

as f is expansive. Therefore Ad_{U_f} restricts to a map between the uniform Roe algebras of X and Y. Its inverse is given by $\mathrm{Ad}_{U_f^{-1}}$, and both of them are injective, henceforth Ad_{U_f} is a $*$-isomorphism between the respective unifrom Roe algebras.

Corollary 3.10. The isomorphism class of the uniform Roe algebra is invariant under bijective coarse equivalences.

One may prove that the Morita-equivalence class of the uniform Roe algebra is invariant under coarse equivalences. The proof reassembles the one for bijective coarse equivalences but uses more technical tools. Here are the main steps:

1. By the theorem of Rieffel, two unital C^*-algebras A, B are Morita-equivalent if and only if they are stable isomorphic, i.e. $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$;
2. Represent the algebra $C_u^*(X) \otimes \mathbb{K}$ on $\ell^2(X) \otimes \ell^2(\mathbb{N})$ in the obvious way;
3. Recall that a coarse equivalence $f\colon X \to Y$ between ulf metric spaces (X,d) and (Y,∂) is uniformly finite-to-one, i.e. $\sup_y |f^{-1}(y)| < \infty$;
4. Assume f is surjective, and define the unitary U_f as follows:
   (a) if $|f^{-1}(y)| = 1$, then set $U_f\delta_{f^{-1}(y)} \otimes \delta_n = \delta_y \otimes \delta_n$;
   (b) if $|f^{-1}(y)| = N$, split $\mathbb{N}$ into N copies of $\mathbb{N}$, and let $f_k\colon \mathbb{N} \to \mathbb{N}$ denote a bijection between $\mathbb{N}$ and the k-th copy of $\mathbb{N}$. Define $U_f\delta_{x_k} \otimes e_n = \delta_y \otimes e_{f_k(n)}$, where $f^{-1}(y) = \{x_1, \ldots, x_N\}$.
5. If f is not surjective, let $\tilde{f}\colon X \to f(X)$ denote the corestriction of f. By coboundedness, there exists a coarse retraction $p\colon Y \to f(X)$, which is a surjective coarse equivalence. Define $U_f = U_p^* U_{\tilde{f}}$;
6. One checks similarly that controlled propagation operators are mapped to control propagation operators. Vise-versa one applies the same technics for $\mathrm{Ad}_{U_f^*}$;

Corollary 3.11. The Morita-equivalence class of uniform Roe algebra is invariant under coarse equivalences.

Generally, there are many C^*-algebras one can associate with a metric space in a similar manner. These algebras are called Roe algebras; the difference between these C^*-algebras is the choice of the Hilbert space on which the potential algebra acts. This choice leads to significant differences in properties that the algebras possess. For example, there are Roe algebras whose isomorphism class is an invariant of coarse-equivalences.
The rigidity problem for uniform Roe algebras asks two questions converse to the above corrolaries:

1. If the uniform Roe algebras of (X, d) and (Y, ∂) are Morita-equivalent, is it true that the underlying spaces are coarsely equivalent?

2. If the uniform Roe algebras of (X, d) and (Y, ∂) are isomorphic, is it true that the underlying spaces are bijectively coarsely equivalent?

These questions were asked by Jhon Roe to Ján Špakula and by Guoliang Yu to Rufus Willett independently to understand better the failure of the coarse Baum–Connes conjecture.

Theorem 3.12 (Špakula, Willett)**.** For two metric spaces (X, d) and (Y, ∂) that satisfy property A the following holds:

1. If $C_u^*(X) \cong C_u^*(Y)$, then (X, d) and (Y, ∂) are bijectively coarsely equivalent;

2. If $C_u^*(X) \sim_{\text{Morita}} C_u^*(Y)$, then (X, d) and (Y, ∂) are coarsely equivalent;

To prove such a theorem, one should first note that any $*$-isomorphism (stable $*$-isomorphism) ϕ between unifrom Roe algebras is spatially implemented (i.e. for some unitary U between the respective Hilbert spaces one has $\phi = \mathrm{Ad}_U$). Then, one may consider the following subset of $Y \times X$:

$$f_\delta = \{(y, x) \mid |\langle U\delta_x, \delta_y\rangle| > \delta\}, \quad \delta > 0.$$

It can be proven that for a carefully picked δ, the set f_δ is a map and a coarse equivalence. We will only sketch the proof that any $*$-isomorphism is spatially implemented.

Lemma 3.13. Let (X, d) and (Y, ∂) be metric spaces, $\phi\colon C_u^*(X) \to C_u^*(Y)$ be a $*$-isomorphism, then ϕ is spatially implemented by some unitary $U\colon \ell^2(X) \to \ell^2(Y)$.

Sketch of proof. Note that the C^*-subalgebra of compact operators on $\ell^2(X)$ is the unique minimal ideal of $C_u^*(X)$. Indeed, it is an ideal in $\mathcal{B}(\ell^2(X))$, hence it is an ideal in $C_u^*(X)$. If J is another ideal in $C_u^*(X)$, then $J \cap \mathbb{K}(\ell^2(X))$ is an ideal in $C_u^*(X)$ and an ideal in $\mathbb{K}(\ell^2(X))$, but $\mathbb{K}(\ell^2(X))$ is simple. Therefore either $J \supset \mathbb{K}(\ell^2(X))$, or $J \cap \mathbb{K}(\ell^2(X)) = \{0\}$. If the second option holds, then $aJ = 0$, for every finite-dimensional operator in $\mathcal{B}(\ell^2(X))$, hence $J = \{0\}$. Since ϕ is a $*$-isomorphism it restricts to a $*$-isomorphism between the unique minimal ideals of $C_u^*(X)$ and $C_u^*(Y)$, hence it restricts to a $*$-isomorphism $\phi\colon \mathbb{K}(\ell^2(X)) \to \mathbb{K}(\ell^2(Y))$. Any such $*$-isomorphism is implemented by a unitary $U\colon \ell^2(X) \to \ell^2(Y)$ (as it is just a change of the basis). It remains to prove that $\phi = \mathrm{Ad}_U$ on the whole algebra. This is done by showing that ϕ viewed as a map

$$\phi\colon \{T \in C_u^*(X) \mid \mathrm{prop}(T) \le R\} \to \mathcal{B}(\ell^2(Y))$$

is SOT-continious. As $\mathbb{K}(\ell^2(X))$ is SOT-dense in $\mathcal{B}(\ell^2(X))$ it follows that ϕ conincides with Ad_U on all finite propagation operators. Hence, ϕ is equal to Ad_U as finite propagation operators are norm-closed. ☐

Property A is crucial for the proof of Theorem 3.12, and it was an open problem whether the rigidity phenomena holds in general. As Alessandro told me, the following theorem was proven during the COVID-19 lockdown via Zoom.

Theorem 3.14 (Baudier, Braga, Farah, Khukhro, Vignati, Willett, 2022)**.** Let (X, d) and (Y, ∂) be uniformly locally finite spaces, then (X, d) and (Y, ∂) are coarsely equivalent if and only if $C_u^*(X)$ and $C_u^*(Y)$ are Morita-equivalent.

In particular, the Morita-equivalence class of the uniform Roe algebra is a complete invariant of coarse equivalences. The question of whether having $*$-isomorphic Roe algebras implies being bijectively coarsely equivalent remains open. It is proven that the rigidity phenomena holds in the following cases:

1. At least one of the spaces has property A;

2. At least one of the spaces is nonamenable;

3. Both spaces are expander graphs.

Recall that all amenable groups have property A; therefore, the rigidity problem for groups is completely solved.

Corollary 3.15. Let G and H be finitely generated groups endowed with right invariant metrics from their Cayley graphs. Then

1. G and H are quasi-isometric if and only if their Roe algebras are Morita-equivalent;
2. G and H are biLipschitz equivalent if and only if their uniform Roe algebras are isomorphic.

Thank you for your attention!